Bipartite network characterization of fluid flows and its relation with the classical Lyapunov exponent

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Mixing and dispersion between different regions of a fluid domain have been characterized during the last decades by the Lagrangian description of motion and dynamical systems theory. A classical measure quantifying dispersion is the Lyapunov exponent, which gives the stretching rate of infinitesimal material line under the evolution of the flow. From the flow-map approach (the flow map is the function that maps initial to final conditions under time evolution during a time τ) we compute the Finite Time Lyapunov Exponent (FTLE) as

$$\lambda(x_0, t_0, \tau) = \frac{1}{2\tau} \log(\Lambda_{\text{max}}), \tag{1}$$

where Λ_{max} is the maximum eigenvalue of the Cauchy-Green strain tensor constructed from the Jacobian matrix of the flow map.

On the other hand, we can also characterize mixing and dispersion between regions from network-theory tools. In Ref. [1] a formalism was developed in which the fluid domain is coarse-grained and a discrete version of the Perron-Frobenious operator, giving a quantification of the amount of fluid going from one fluid box to another one, is used to define link strengths between these fluid boxes, thus defining a temporal, weighted and directed flow network.

Here we develop the above formalism to characterize transport between two layers in a three-dimensional fluid flow (a possible application is the quantification of transport between surface and bottom of the ocean). For our purpose, we construct the flow map for the three dimensional incompressible Arnold-Beltrami-Childress flow model whose velocity field is defined as

$$\dot{x} = A \sin z + C \cos y, \tag{2a}$$
$$\dot{y} = B \sin x + A \cos z, \tag{2b}$$
$$\dot{z} = C \sin y + B \cos x. \tag{2c}$$

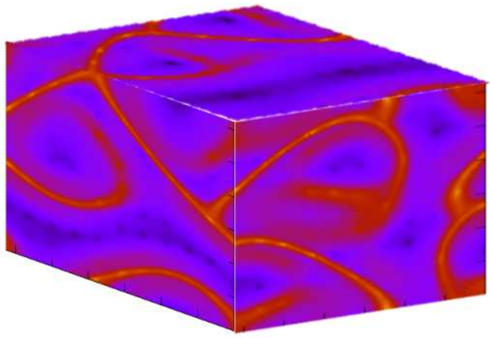

Fig. 1. FTLE of the three dimensional ABC flow with periodic boundary conditions for the integration time $\tau = 3$, and $A = \sqrt{3}$, $B = \sqrt{2}$, $C = 1$

We construct a bipartite network with links describing the amount of fluid transported between fluid boxes located at two horizontal layers embedded into the fluid domain. We then characterize this bipartite network, find similarities between the Lyapunov exponent and network properties (see the three-dimensional FTLE field for this flow in Fig. 1), and extract coherent structures generalizing the ideas in Refs. [1, 2].

[1] E. Ser-Giacomi, V. Rossi, C. López, E. Hernández-García, Flow networks: A characterization of geophysical fluid transport, Chaos **25**, 036404 (2015).

[2] E. Ser-Giacomi, V. Rodríguez-Méndez, C. López, E. Hernández-García, Lagrangian Flow Network approach to an open flow model, Eur. Phys. J.-Spec. Top. **226**. 2057-2068 (2017).