Synchronization invariance under network structural transformations

Lluís Arola-Fernández[1], Albert Díaz-Guilera[2,3], and Alex Arenas[1]
[1]Departament d'Enginyeria Informàtica i Matemàtiques, Universitat Rovira i Virgili, 43007 Tarragona, Spain
[2]Departament de Física Fonamental, Universitat de Barcelona, Martí i Franquès 1, 08028 Barcelona, Spain
[3]Universitat de Barcelona Institute for Complex Systems (UBICS), Barcelona, Spain

The study of dynamical processes running on top of complex networks has become a central issue in many research fields, ranging from the microscopic realm of genes and neurones to large technological and social systems. However, many times the information we can accede to about the actual topology of interactions is partially incomplete. Moreover, given that the only reflection of the dynamics on networks is usually a certain macroscopic observable, it turns out that many topologies are compatible with the same dynamical output, raising the problem of multi-valuation. Following this perspective, we analyze the relation between function and structure in a novel mapping problem: given a certain network structure and a dynamical process on top of it, we wonder how to transform the network into a different structural connectivity so that the collective behavior of the system remains invariant.

To derive the network transformations, we focused on a paradigmatic example of emergent phenomena, the synchronization of coupled phase oscillators in the Kuramoto Model (KM) [2], which consists of a population of N coupled phase oscillators that evolve in time according to

$$\dot{\theta}_i = \omega_i + K \sum_{j=1}^{N} \lambda_{ij} \sin(\theta_j - \theta_i),\ \forall i\ \in N, \quad (1)$$

where θ_i is phase of the i-oscillator, ω_i its natural frequency, drawn from a probability distribution $g(\omega)$, λ_{ij} are the elements of the coupling matrix $\mathbf{\Lambda}$ that capture the presence of a connection and its intensity and K is a constant coupling strength that scales all the weights. The collective behavior of the KM is described through the complex order parameter $re^{i\Psi(t)} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}$, where the modulus r measures the overall degree of synchrony. Here, we assume that r is the only available observable from measurements, and we look for transformations of $\mathbf{\Lambda}$ that keep r invariant, for any value of the control parameter K. In particular, we study the mapping between two structurally different networks (the target $\mathbf{A}$ and candidate $\mathbf{B}$) of N distinguishable oscillators. We aim to find transformations of the weights (the intensity of the connections) in the candidate network, without altering its structure, in a way that the macroscopic response $r(K)$ is identical to the one in the target network.

Inspired by the derivation of statistical mechanics from information theory as a particular case of statistical inference, see [3], we tackle the mapping as an optimization problem for the unknown weights subject to local structural constraints in the system. In particular, we impose an entropy maximization for the weights distribution subject to a detailed balance that constraint, up to a given order m, the input strengths of the nodes (the sum of incoming connections of the m-neighbors) in both networks. We derive analytical expressions for the weights according to different

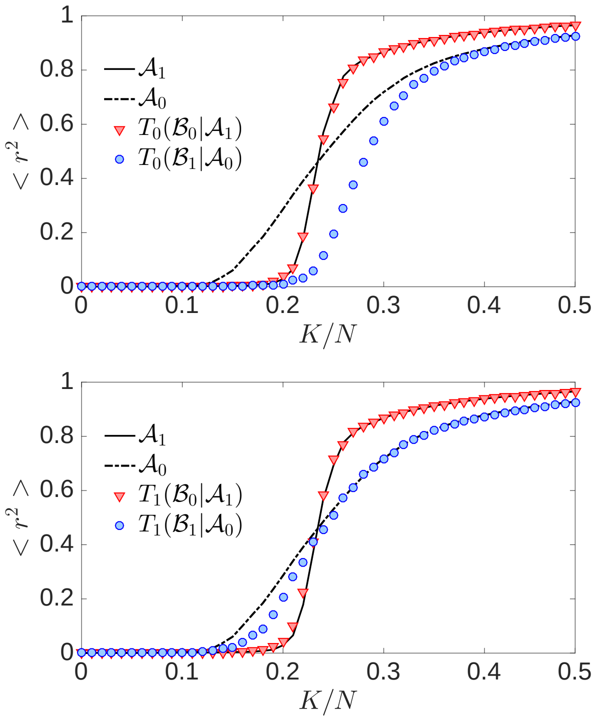

Fig. 1. Synchronization diagrams ($N = 2000$) for target networks A_1 (Erdös-Rényi) and A_0 (power-law), which are unweighted and symmetric, and its respective transformations $T(B_1|A_0)$ and $T(B_0|A_1)$, (top) preserving only zero-order input-strengths and (bottom) preserving also the first-neighbours input-strengths (exploiting more information).

states of available information, and we show that the invariance condition can be achieved even if the mapped networks have very different connectivity patterns or the system is in the non-linear regime. Furthermore, we show that the mapping of homogeneous networks into heterogeneous ones is usually less accurate and requires more -costly- microscopic information than the reverse process, unveiling a symmetry-unbalance phenomenon that emerges from the partial impossibility of preserving the properties of the nodes in the transformation (see Fig.1). The presented formalism can be applied in a wide spectra of existing problems beyond the mapping scenario and provides new analytical insight to tackle real complex scenarios when dealing with uncertainty in the measurements of the underlying connectivity structure.

[1] L. Arola-Fernández, A. Díaz-Guilera, and A. Arenas, Phys. Rev. E **97**, 060301(R) (2018).

[2] A. Arenas, A.Díaz-Guilera, J .Kurths, Y. Moreno, C. Zhou, Phys. Rep. **469**, 93-153 (2008).

[3] E. T. Jaynes, Phys. Rev. **106**, 620-630 (1957).