Emergence of Gaussian statistics as a symmetry far from equilibrium

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Suitable non-equilibrium conditions have been recently shown to allow for *symmetry emergence*, as opposed to *spontaneous symmetry breaking*, in extended systems [1]. A paradigmatic model in statistical physics is the stochastic Burgers equation,

$$\partial_t \phi = \nu \partial_x^2 \phi + \lambda \phi \partial_x \phi + \eta, \tag{1}$$

where η is space-time, white noise. Indeed, Eq. (1) appears in many different contexts [2], see, e.g., Fig. 1. Moreover, Eq. (1) can be generalized to higher dimension, as, e.g., [3]

$$\partial_t \phi = \nu_x \, \partial_x^2 \phi + \nu_y \, \partial_y^2 \phi + \lambda_x \phi \partial_x \phi + \lambda_y \phi \partial_y \phi + \eta, \tag{2}$$

which also generalizes the Hwa-Kardar (HK; $\lambda_y = 0$) equation that describes avalanches in running sandpiles [4]. Furthermore, Burgers equation is strongly related with other important models: The change of variable $\phi = \partial_x h$ transforms the deterministic terms of Eq. (1) into those of the 1D Kardar-Parisi-Zhang (KPZ) equation, another paradigm of contemporary non-equilibrium statistical physics [5].

Both the KPZ and the stochastic Burgers equations exhibit generic scale invariance [6]: The variance W^2 of the field grows up to a saturation value W_{sat}^2 at time t_{sat}, such that $W_{\text{sat}} \sim L^\alpha$ and $t_{\text{sat}} \sim L^z$, where L is the lateral size of the system. Universality classes occur, which are characterized by the values of α, z, and by the statistics of fluctuations; for the 1D KPZ equation, the latter is provided by the Tracy-Widom (TW) distribution [5], whose universal, nonzero skewness manifests the lack of up-down symmetry ($h \leftrightarrow -h$) of the system.

The scaling exponents of Eqs. (1)-(2) have been investigated both analytically [4, 7] and numerically [3, 8]. However, the statistics of the field had not been reported in the literature for the Burgers and Hwa-Kardar equations yet. Due to their nonlinearities, Eqs. (1)-(2) also lack up-down symmetry ($\phi \leftrightarrow -\phi$); hence, fluctuations in ϕ are expected to be skewed and non-Gaussian, as in the KPZ case. However, this seems not to be the case.

In this work [9], we revisit the universality class of the Burgers and the (generalized) HK equations, focusing on the statistics of fluctuations. Remarkably, these turn out to be Gaussian, see Fig. 2. We reach this conclusion from numerical simulations and from dynamic renormalization group calculations of the skewness and kurtosis of the field ϕ.

The scaling exponents of Eqs. (1)-(2) are fixed by the hyperscaling ($2\alpha + d = z_d$) and Galilean ($\alpha + z_d = 1$) scaling relations, induced by non-renormalization of noise and nonlinearity, respectively [3, 4, 7]. Actually, both the Gaussian statistics and these exponent values are *exact* for the *linear* (hence, up-down symmetric) equation

$$\partial_t \hat{\phi} = \left(-\sum_{i=1}^{d} |k_i|^{z_d} \right) \hat{\phi} + \hat{\eta}, \tag{3}$$

where hat is space Fourier transform and $\mathbf{k}$ is wave-vector.

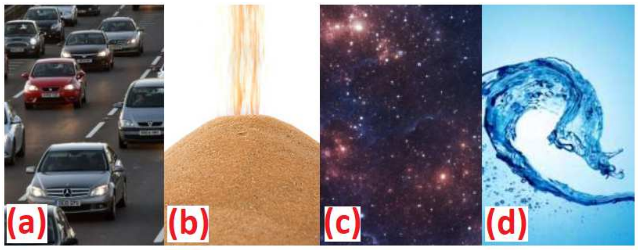

Fig. 1. Systems described by Eqs. (1)-(2) and meaning of ϕ [2, 3, 4]: (a) traffic models (vehicle density), (c) avalanche dynamics (pile height), (b) cosmology (mass density in the early universe), and (d) turbulence (fluid velocity).

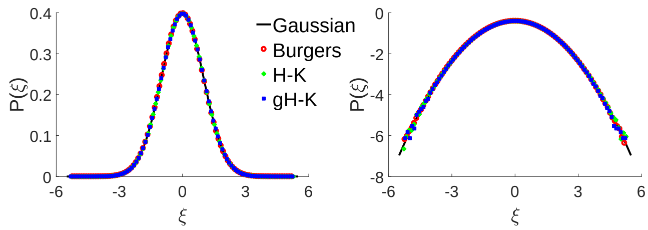

Fig. 2. Normalized fluctuation histogram from numerical simulations of the Burgers and (generalized) Hwa-Kardar equations, Eqs. (1)-(2). Here, $\xi = (\phi - \bar{\phi})/\text{Var}(\phi)$.

Overall, the up-down symmetry, notably absent from Eqs. (1)-(2) themselves, *emerges* at the critical point which governs their large-scale behavior, in the form of up-down-symmetric, Gaussian fluctuations. Indeed, Gaussian statistics can be expected far from equilibrium, even for systems which are closely related with non-Gaussian, KPZ universality.

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