An equation for biased diffusion in uniformly growing domains

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Anomalous diffusion models are a very useful tool to describe key features of biological systems where non-Fickian transport is at play. In this context, there are numerous examples where systematic forces influence the particle spreading. The effect of such forces can be accounted for by introducing a bias term in the corresponding transport equation. A typical example is the action of a chemotactic gradient sensed by a collection of bacteria.

While the effect of drift terms on anomalous diffusion processes is well studied in the case of static domains, in a wide variety of biological systems the dissemination of the particles takes place while the medium itself grows at a non-negligible rate. Examples include proliferative tissue growth and the formation of pigmentation patterns in growing organisms. Considering the problem of biased anomalous transport in growing domains is thus of great practical importance.

In the above context, we shall consider the celebrated Continuous-Time Random-Walk model (CTRW), which is well characterized in the case of a one-dimensional static domain [1]. In this model, particles perform instantaneous jumps interrupted by waiting times which follow the probability density function (PDF) $\varphi(t)$. In its simplest version, the single-jump displacement is considered to be uncoupled from the waiting-time PDF and given by the PDF $\lambda^*(y)$. When λ^* has a finite variance Σ^2, the system displays subdiffusion when the Laplace transformed waiting-time PDF behaves as $\tilde{\varphi}(s) \sim 1 - \tau^\alpha s^\alpha$ for $s \to 0$, where $0 < \alpha < 1$. The case $\alpha = 1$ gives rise to normal diffusion.

But, what happens if the medium expands? In this case, the purely diffusive motion is influenced by an additional drift arising from the stretching of physical space. Recent works [2, 3] show that this problem is amenable to analytical treatment by switching to so-called comoving coordinates x. The latter are defined as the projections of the physical points y on the initial domain. In the case of a uniform expansion, the relation between both sets of coordinates is straightforward, i.e., $y = a(t)x$. In comoving space, displacements are thus shortened by the inverse scale factor $1/a(t)$, implying that $\lambda^*(y)$ must be replaced by $\lambda(x,t) = a(t)\lambda^*(a(t)x)$.

The introduction of an external force field, $F^*(y,t)$, results in an asymmetric jump-length distribution. In the simplest case, there is a linear dependence between the force and the jump asymmetry [1]. In comoving coordinates, this force is expressed as $F(x,t) = F^*(a(t)x,t)$. In the long-time limit, the above CTRW scheme leads to the following fractional diffusion equation

$$\frac{\partial W(x,t)}{\partial t} = \frac{K_\alpha}{a^2(t)} \frac{\partial^2}{\partial x^2} {}_0D_t^{1-\alpha} W(x,t) - \frac{1}{\xi_\alpha a(t)} \frac{\partial}{\partial x} \left[F(x,t) {}_0D_t^{1-\alpha} W(x,t) \right], \tag{1}$$

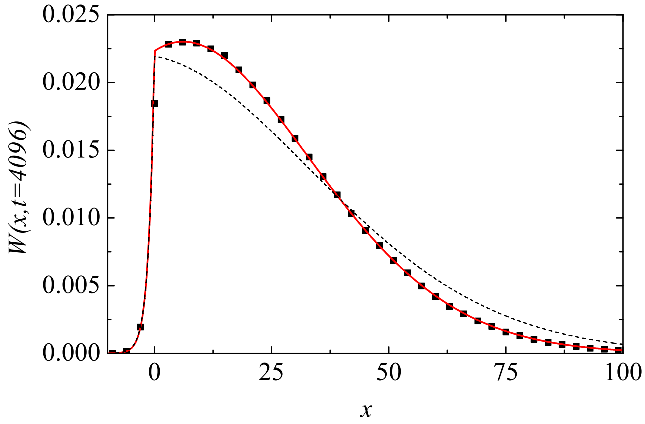

Fig. 1. Comoving propagator at time $t = 4096$ for subdiffusive particles with $\alpha = 1/2$ and $K_\alpha = 1/2$ drifted by a constant force $F_0 = \xi_\alpha/\sqrt{2\pi}$ and an exponential expansion $a(t) = \exp(Ht)$ with $H = 10^{-4}$. The solid line draws a numerical integration of Eq. (1) using the Crank-Nicolson method [4] with time and spatial discretization of $1/10$ units. The squares are the simulation results after 10^6 runs. The dashed line plots the same curve for $H = 0$. It has been obtained by means of the subordination technique [1].

where $K_\alpha = \Sigma^2/(2\tau^\alpha)$ is the diffusivity and ξ_α denotes the generalized friction constant. The operator ${}_0D_t^{1-\alpha} f(t)$ is defined as the inverse Laplace transform of $s^{1-\alpha}\tilde{f}(s)$.

The free boundary solution of Eq. (1) for Brownian particles subjected to a constant force F_0 and to the initial condition $W(x,0) = \delta(x)$ is a shifted Gaussian with time-dependent first moment $F_0 \int_0^t du\; a^{-1}(u)/\xi_1$ and variance $2K_1 \int_0^t du\; a^{-2}(u)$ [2]. However, the numerical integration of Eq. (1) reveals that the propagator is non-symmetric for subdiffusive CTRWs and it may not be represented by a suitable coordinate rescaling of its counterpart for the static case (see Fig. 1). Numerical simulations based on the Monte-Carlo algorithm confirm this finding.

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