First-passage distributions for the one-dimensional Fokker-Planck equation

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Imagine we know the state of a system, say the position of a diffusive particle, at an arbitrary initial time, and we let the system evolve, under whatever stochastic dynamics. In these conditions, one might inquire into the time needed for the system to reach another particular state or to return to the original state for the first time. The concepts of first-passage (FP) and first-return (FR) times are intuitively related to these scenarios. They provide valuable information of the temporal properties of the system and, in turn, are relatively easy to obtain experimentally or by means of simulations. A consequence of this has been their immediate applicability in a myriad of problems within statistical physics and beyond: spreading of diseases [1], animal or human movement [2], neuron firing dynamics [3], diffusion controlled reactions [4], controlled kinetics [5], or renewal and non-renewal systems [6], ... At an analytical level, many techniques have been developed to compute the FP and FR times, their associated probability density functions and the moments of these distributions [7, 8]. However, general formulations are scarce, since one finds a large variability from one problem to another: the geometry on which the system is embedded, the nature of the boundary conditions, the continuum or discrete character of the dynamics, and, specially, the microscopical rules driving the evolution of the system. In simple cases, though, this is a solved problem. For example, if the dynamics is described by means of a Fokker-Planck equation, one can relate its solution with the FP and FR distributions. Can we infer relevant information about these distributions when the solution is unknown?

We tackle this and related questions along the article in this work [9]. Specifically, we explore the FP and FR time distributions of the large family of models represented by the one dimensional Fokker-Planck equation in bounded domains, with state dependent drift and diffusion terms. We prove analytically that the distributions of all these models may decay as a power law whose exponents can take different values, depending on some conditions involving the diffusion and drift terms as well as the initial and final states. When the diffusion coefficient is positive and the drift term is bounded, like the random walk, both distributions obey a universal law that exhibits a power-law decay of exponent $-3/2$ for low and intermediate times. We also discuss the influence of an absorbing state, characterized by a vanishing diffusion coefficient and/or a diverging drift term. Remarkably, the random walk exponent is still found, as far as the departure and arrival regions are far enough from the absorbing state. Close enough to the absorbing point, though, new universal laws emerge, their particular properties depending on the behavior of the diffusion and drift. We focus on the case of a diffusion term vanishing linearly. In this case, FP and FR distributions show a new *universality class*, characterized by the eventual presence of a power law with exponent -2. As illustration of the predictions of our general theory, we systematically study, both analytically and computationally, different models of increasing complexity. The

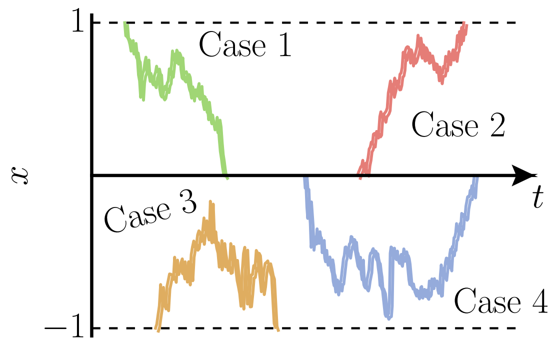

Fig. 1. Types of trajectories that we study. In cases 1 and 2 we compute the first-passage time distribution from the border of the domain to its center, and viceversa. In cases 3 and 4, we compute the first-return time distribution, departing from and arriving to the same border of the domain, and departing from and arriving to the center of the interval.

models can be mapped into a family of two-parameter models which include the random walk, the Ornstein-Uhlenbeck process, the voter model, and two noisy variations of the latter. We use them as representative examples of the different universality classes and the rich behaviour of the temporal properties of the FP and FR distributions.

The impact of the work is twofold. First, it permits to connect under a general lens some independent results that have been separately obtained. Second, it allows an immediate identification between the power law decays and the type of dynamics driving the system. Therefore, our results offer the possibility to unveil the main properties of a process (e.g., presence of absorbing states, nature of the physical boundaries, ...) just by looking at the first-passage and first-return distributions.

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