On a graph-theoretical structure of real numbers

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In this work, we study properties of real numbers through a set of graphs named *Farey graphs*, which we show are in bijection with real numbers in $[0, 1]$. The Farey graphs can be navigated by an operator $\mathcal{R}$. This operator induces a dynamics and we make a classification of dynamical attractors (fixed points, periodic or aperiodic orbits and chaos) which has a correspondence in the real numbers. Furthermore, we can define an entropy on Farey graphs, and its maximization connects with the previous dynamical classification.

The Farey sequence of order n is the ordered set of irreducible fractions between $[0, 1]$ whose denominators do not exceed n. The Farey sequence $\mathcal{F}_n$ has a representation called *Farey Tree* (see Fig. 1). When $n \to \infty$, the Farey sequences are the real numbers between $[0, 1]$.

The set of Farey graphs is constructed recursively using a initial graph (two nodes joined by a link) and an inner operation named concatenation (see Fig. 2). We prove that there exists a bijection between the Farey graphs and Farey fractions, ie, between the interval $[0, 1]$. This implies that the Farey graphs have an order and we can represent them in a tree named *Farey Graph Tree*.

The operator $\mathcal{R}$ is a map of the set of Farey graphs into itself removing the nodes with degree $k = 2$ and merging its two incident edges into a single edge. $\mathcal{R}$ has an algebraically equivalent operator in real numbers: the operator $T : [0, 1] \to [0, 1]$

$$T(\omega) = \begin{cases} \frac{\omega}{1-\omega} & \text{if } \omega \leq 1/2 \\ 1 - \frac{1-\omega}{\omega} & \text{if } x > 1/2 \end{cases} . \qquad (1)$$

The dynamics of this operator induces a classification of real numbers into families:

1. **Fixed Points:** The point $\omega = 0$ is an attractor for all rational initial conditions.

2. **Unstable Periodic Orbits:** The quadratic irrationals belong to a cycle, i.e, they verify:

   $\exists\, m \geq 2 :\ T^{(m)}(\omega) = \omega$ iff ω is quadratic irrational.

3. **Chaos:** All other initials conditions (e.g, non-quadratic algebraic irrational and trascendental numbers).

Finally, we are interested in a particular graph entropy over the degree distribution $P(k)$. We compute this entropy for all graphs with at least 1000 nodes (see Fig. 3). We prove that the most Farey-entropic graph corresponds to the fractional part of the Golden ratio. Other form a periodic orbit and they correspond to the quadratic irrationals.

[1] R. L. Graham, D. E. Knuth, and O. Patashnik, *Concrete Mathematics* (Addison-Wesley, 1989/1994).

[2] B. Luque, F. J. Ballesteros, A. M. Nunez, and A. Robledo, Quasiperiodic graphs: Structural design, scaling and entropic properties, J. Nonlinear Sci. **23**, 335-342 (2013).

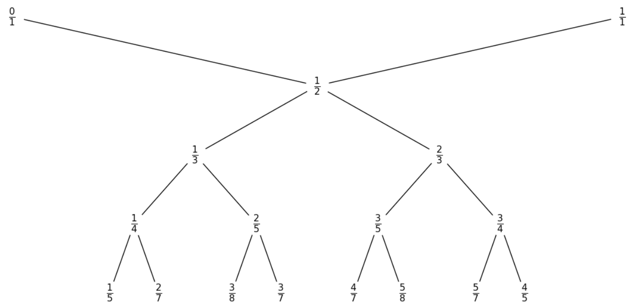

Fig. 1. The first five layers of Farey Tree.

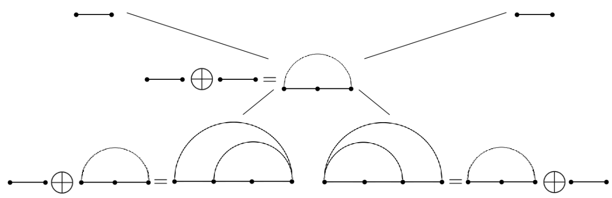

Fig. 2. An illustration of the concatenation operation.

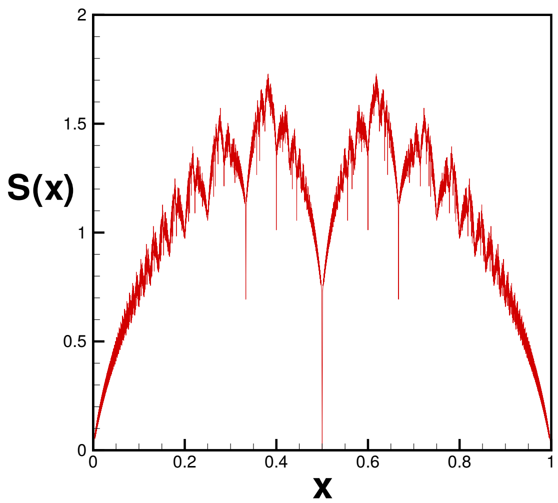

Fig. 3. Number entropy $h(\omega) = -\sum P(k) \log P(k)$ for $G = G_\omega$.