Relaxation time of the global order parameter on multiplex networks: The role of interlayer coupling in Kuramoto oscillators

A. Allen-Perkins[1,2], T. A. de Assis[1], J. M. Pastor[2], and R. F. S. Andrade[1]
[1]Instituto de Física, Universidade Federal da Bahia, 40210-210 Salvador, Brazil
[2]Complex System Group, Universidad Politécnica de Madrid, 28040 Madrid, Spain

This work presents a new formalism to study analytically the time scales of the global order parameter and the interlayer synchronization of coupled Kuramoto oscillators on multiplex networks [1].

We consider an undirected multiplex $\mathcal{M}$ with $M = 2$ layers G^{α}, $1 \leq \alpha \leq M$, where each layer contains N nodes identified by x_n^{α}, $1 \leq n \leq N$ (see Fig. 1). The oscillator in each node x_n^{α} of the layer G^{α} is characterized by its phase θ_n^{α}, whose dynamics is described by

$$\dot{\theta}_n^{\alpha} = \Omega_n^{\alpha} + \lambda^{\alpha} \sum_{x_m^{\alpha} \in G^{\alpha}} w_{nm}^{\alpha} \sin(\theta_m^{\alpha} - \theta_n^{\alpha}) + \lambda^{12} w_{nn}^{12} \sin(\theta_n^{\beta} - \theta_n^{\alpha}). \quad (1)$$

Here, Ω_n^{α} is the natural frequency of the oscillator x_n^{α}, λ^{α} and λ^{12} are the coupling strength of the layer α and of the interlayer 12, respectively, w_{nm}^{α} is the weight of the connection between the nodes x_n^{α} and x_m^{α}, and w_{nn}^{12} is the weight of the connection between the nodes x_n^{α} and x_n^{β}.

For two-layer multiplexes with an initially high degree of synchronization in each layer (see Fig. 2), the difference between the average phases in each layer, denoted by $\Delta = \psi^1 - \psi^2$, with

$$\psi^{\alpha}(t) = \mathrm{Arg}\left(\frac{1}{N} \sum_{x_n^{\alpha} \in G^{\alpha}} e^{i\theta_n^{\alpha}(t)}\right), \quad (2)$$

is analyzed from two different perspectives: the spectral analysis and the nonlinear Kuramoto model.

Both viewpoints confirm that the time scales of the global order parameter r, being

$$r(t) = \frac{1}{2N} \left| \sum_{x_n^{\alpha} \in \mathcal{M}} e^{i\theta_n^{\alpha}(t)} \right| \approx \left| \cos\left(\frac{\Delta}{2}\right) \right|, \quad (3)$$

and of the interlayer synchronization Δ are inversely proportional to the interlayer coupling strength, λ^{12}. Thus, increasing the interlayer coupling always shortens the transient regimes of both r and Δ

The analytical results show that the convergence of the global order parameter is faster than the interlayer synchronization, and the latter is generally faster than the global synchronization of the multiplex. The formalism also outlines the effects of frequencies on the difference between the average phases of each layer, and it identifies the conditions for an oscillatory behavior. Computer simulations are in fairly good agreement with the analytical findings, and they reveal that the time scale of the global order parameter is half the size of the time scale of the multiplex, if not smaller.

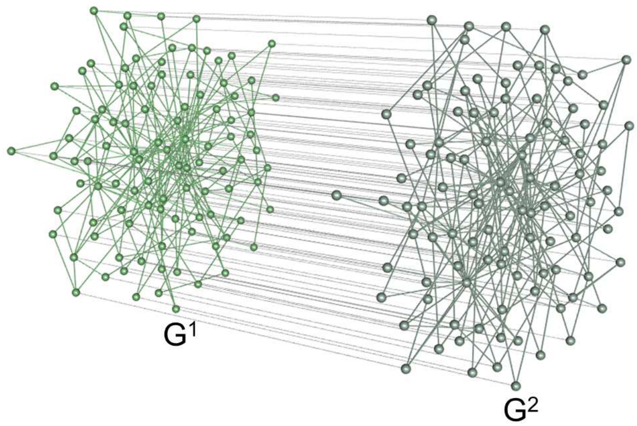

Fig. 1. Example of an undirected multiplex network with two layers, G^1 and G^2. Taken from Ref. [1].

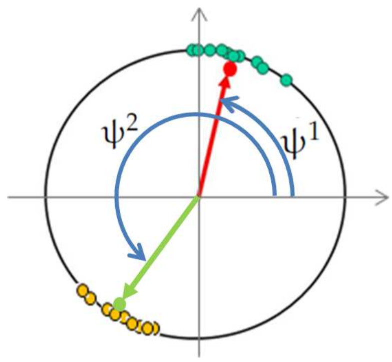

Fig. 2. Example of an undirected multiplex network with two layers and a high degree of synchronization in each layer (turquoise and yellow dots for, respectively, the phases of the oscilators in G^1 and G^2).

[1] A. Allen-Perkins, T. A. de Assis, J. M. Pastor, and R. F. S. Andrade, Relaxation time of the global order parameter on multiplex networks: The role of interlayer coupling in Kuramoto oscillators, Phys. Rev. E **96**, 042312 (2017).